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Notes on a SQCD-like plasma dual and holographic renormalization

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Abstract: We study the thermodynamics and the jet quenching parameter of a black hole solution dual to a SQCD-like plasma which includes the backreaction of fundamental flavors. The free energy is calculated in several ways, including some recently proposed holographic renormalization prescriptions. The validity of the latter is confirmed by the consistency with the other methods. The resulting thermodynamic properties are similar to the Little String Theory ones: the temperature is fixed at the Hagedorn value and the free energy is vanishing. Finally, an accurate analysis of the relevant string configurations shows that the jet quenching parameter is zero in this model, in agreement with previous findings.

Keywords: AdS-CFT Correspondence, Black Holes in String Theory, Gauge-gravity correspondence.

Contents

1. Introduction and conclusion

In the context of the string/gauge theory correspondence [1], the study of theories at finite temperature is recently attracting renewed interest. In fact, from the first works on the subject [2, 3], the investigation of properties of thermal field theories and the thermodynamics of the dual black holes have revealed an exciting correspondence between these two previously independent topics. In particular, the calculation of quantities such as the free energy, entropy or energy density can be performed in the gravity side, providing results for the thermodynamics of the dual field theories. Since in performing such computations the observables typically turn out to be infinite, one has to employ renormalization techniques in order to extract the meaningful physical informations. One such procedure, which goes under the name of "holographic renormalization", is particularly interesting, since it provides a way of renormalizing infinities in gravity computations, under the requirement that the dual observables in field theory must be finite [4–6], thus connecting the counterterm renormalization in field theory with a specific subtraction mechanism on the gravity side.

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Moreover, recently the string/gauge theory correspondence for thermal systems has become of great importance because of the evidence that string theory may provide a useful tool to study some properties of Quark Gluon Plasmas (QGPs) as the one experimentally produced at RHIC [7]. Relevant examples include hydrodynamic features, jet quenching phenomena, screening properties and photoproduction (see [8]–[16] and the references to these works).

Since the string models usually employed for the latter task do not account for the backreaction of fundamental flavors, with the exception of [17], it is clearly of interest to try and study what are the effects of the "quarks" on the stringy predictions for the QGP. In this note we consider the only known ten dimensional black hole solution dual to a four dimensional, finite temperature field theory that includes the backreaction of many flavor degrees of freedom [18]. The zero temperature solution is dual to a $\mathcal{N}=1$ SQCD theory with a superpotential, with number of flavors N_f being the double of the number of colors N_c, $N_f = 2N_c$. The solution corresponds to wrapped "color" D5-branes and smeared "flavor" D5-branes.

After the introduction of the solution in section 2, we study its thermodynamic properties in section 3. The results are very similar to the Little String Theory ones, consistently with the fact that the solution comes from D5-branes. The temperature of the black hole is fixed at the Hagedorn value and it is independent on the horizon size. Thus, the free energy density turns out to be exactly zero in the gravity approximation: the energy density is equal to the entropy density times the temperature.

We perform such analysis in three ways. As a first route, we renormalize the infinities in the gravity computation of the free energy by subtracting the analogous contributions of a reference background, namely the zero temperature solution. Alternatively, we separately calculate the energy (by subtracting the reference background value) and the entropy density. Finally, we employ very simple holographic renormalization prescriptions using two different counterterms, one for the ordinary gravity plus surface terms and one for the flavor terms. While the counterterms for the gravity plus surface contributions are fairly standard, the ones for the flavor terms are in principle on a less solid ground, since they have been much less tested.

In fact, in the model at hand the flavor contribution is included by sourcing the supergravity fields with the Born-Infeld action for the smeared D5-branes. In the literature there exist formulas for the counter-terms for *probe* flavor branes [6, 19, 20]. We show that the latter are sufficient to account for the renormalization of the flavor contribution also in the case where the full backreaction of the flavor branes is taken into account. The latter result is not a-priori guaranteed, and the situation could have also been worsened by the smearing procedure. Instead, we find that the using the two counterterms proposed in [20] gives a completely consistent picture of the thermodynamics of the backreacted system. Let us also point out that these two simple counterterms give in the present case a unique renormalization prescription, thus providing a complementary approach to holographic renormalization to the one employed for example in [21] for the Little String Theory case (we briefly review the LST case in our setting in section 3.1).

Finally, in a rather independent part of the paper (section 4), we come to the issue of

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the evaluation of the jet quenching parameter $\hat{q}$. The latter was computed in the model at hand in [17] using the string configuration proposed in [9], with an unexpected vanishing result, $\hat{q}$=0. Given this behavior and considering the proposal made in [22] of using a different string solution for the calculation of $\hat{q}$, we perform an accurate analysis of the configurations that could be relevant for the purpose.[1] We find indications that the only solution that gives a meaningful result (for the action employed for the calculation of $\hat{q}$) gives a vanishing jet quenching parameter, in agreement with the finding in [17].

2. Introducing the black hole

We shall deal with a background which is the thermal deformation of the dual to a $\mathcal{N}=1$ SQCD with a superpotential, coupled to KK adjoint matter [18]. The corresponding black hole is known only for the case $N_f = 2N_c$. Otherwise stated we shall work in the Einstein frame and with $\alpha' = 1$. The metric reads

$$\begin{aligned} ds_0^2 &= e^{(\phi_0+r)/2}\Bigg[d\vec{x}_4^2 + N_c\bigg(dr^2 + \frac{1}{\xi}(d\theta^2+\sin^2\theta d\varphi^2) + \frac{1}{4-\xi}(d\bar\theta^2+\sin^2\bar\theta d\bar\varphi^2) \\ &\quad + \frac{1}{4}(d\psi+\cos\theta d\varphi+\cos\bar\theta d\bar\varphi)^2\bigg)\Bigg], \end{aligned} \tag{2.1}$$

for the “zero temperature” background and

$$\begin{aligned} ds_T^2 &= e^{(\phi_0+r)/2}\Bigg[-\mathcal{F}dt^2 + d\vec{x}_3^2 + N_c\bigg(\frac{1}{\mathcal{F}}dr^2 + \frac{1}{\xi}(d\theta^2+\sin^2\theta d\varphi^2) \\ &\quad + \frac{1}{4-\xi}(d\bar\theta^2+\sin^2\bar\theta d\bar\varphi^2) + \frac{1}{4}(d\psi+\cos\theta d\varphi+\cos\bar\theta d\bar\varphi)^2\bigg)\Bigg], \qquad \mathcal{F} = 1-e^{(r_0-r)}, \end{aligned} \tag{2.2}$$

for the thermal one. The angles run in $0 \le \theta(\bar\theta) \le \pi,\ 0 \le \varphi(\bar\varphi) \le 2\pi,\ 0 \le \psi \le 4\pi$. The horizon of the thermal solution is located at r_0; $t,\ x_i$ are dimensional coordinates, while the rest are dimensionless; ξ is a free parameter that varies in the range $0 < \xi < 4$. The backgrounds include also a dilaton and a three form field which are identical in both cases[2]

$$e^{\phi} = e^{\phi_0+r}\,, \tag{2.3}$$

$$F_{(3)} = -\frac{N_c}{4}\left(\sin\bar\theta d\bar\theta\wedge d\bar\varphi + \sin\theta d\theta\wedge d\varphi\right)\wedge\left(d\psi+\cos\theta d\varphi+\cos\bar\theta d\bar\varphi\right). \tag{2.4}$$

These backgrounds are solutions of the Type IIB supergravity equations of motion for N_c color D5-branes wrapped on a two-sphere, plus a DBI source to account for the backreaction of N_f D5 flavor branes. The latter are aligned along r, ψ and the Minkowski coordinates, and they are smeared in the other directions. Being (2.1), (2.2) (decoupled) D5-brane solutions, they are asymptotic for large radius to (compactified) Little String Theory backgrounds. As such, the black hole shares with the thermal LST solution the main feature that its temperature does not depend on r_0,

$$T = \frac{1}{2\pi\sqrt{N_c}}\,. \tag{2.5}$$

[1]We thank F. Bigazzi for pointing out the possible relevance of this analysis.

[2]Note that there is a typo in equation (4.23) of [18], see eq. (3.18) of the same paper.

As a consequence, its free energy is expected to vanish if the usual thermodynamic relations are still valid for this system. This is indeed the case, as we will explicitly compute in the following. Finally, the functional form of the scalar correlator agrees with that presented in [23] for LST.

3. Thermodynamics and holographic renormalization

In this section we will study the thermodynamics of the above model. As already stressed, being the latter the only available critical model describing a finite temperature field theory with backreaction of the fundamental flavors, its thermodynamic analysis is surely worthwhile.

Moreover, it turns out to be a quite useful arena where to study the holographic procedures in the presence of the flavor branes. In fact, as usual when one evaluates thermodynamic quantities, such as the free energy or the energy density, one deals with infinities. There exist standard procedures in order to extract anyway the relevant informations from supergravity, as measuring them with respect to a reference background or by a subtraction mechanism known as holographic renormalization. But in the case at hand the system does not include only the supergravity action — the latter is indeed coupled to the (smeared) DBI source for the flavor branes. Thus, it is not a-priori clear whether the known procedures for supergravity can be applied to this case. We will instead show that both methods give (the same) meaningful answers for the thermodynamics of the system at hand.

Before coming to the backreacted $\mathcal{N}=1$ SQCD model in section 3.2, we will review some of the underlying aspects one finds in near-extremal NS5-branes, where one does not deal with the flavor contribution.

3.1 Preliminaries: thermodynamics of LST

One of the reasons the LST model is interesting is the fact that it is a non-gravitational theory believed to be dual to a certain string theory background. In the decoupling limit of N coincident NS5-branes the string length l_s is kept fixed while the string coupling g_s goes to zero. In that precise limit the theory reduces to Little String Theory, or more precisely to $(2,0)$ LST for type IIA NS5-branes and $(1,1)$ LST for type IIB NS5-branes [24]. We are interested in the non-extremal case. The decoupling limit is achieved by keeping l_s fixed and taking g_s to zero while the energy above extremality is fixed. The throat geometry described by this setting is [25]

$$ds^2 = e^{-\phi/2}\left[\left(1-\frac{z_0^2}{z^2}\right)dx_1^2+\sum_{j=2}^{6}dx_j^2+N\left(\frac{dz^2}{z^2-z_0^2}+d\Omega_3^2\right)\right], \qquad e^{2\phi}=\frac{N}{z^2}. \tag{3.1}$$

The extremal configuration is obtained by the limit $z_0\to 0$. In the latter limit, (3.1) represents a five-brane whose world-volume is $\mathbf{R}^6$. The string propagation in this geometry should correspond to an exact conformal field theory [26]

$$\mathbf{R}^{5,1}\times\mathbf{R}_\phi\times \mathrm{SU}(2)_N\,. \tag{3.2}$$

In addition to the previous fields there is a NS-NS $H_{(3)}$ form along the S^3, $H_{(3)} = 2N\epsilon_3$. The geometry (3.1) is regular as long as $z_0 \neq 0$ and (in order not to develop a conical singularity) the period of the Euclidean time is chosen as

$$\beta = 2\pi\sqrt{N}\,. \tag{3.3}$$

Notice that this value is *fixed* and independent of the black hole radius, and leads to a complete degenerate thermodynamical phase space [27]. It corresponds to the Hagedorn temperature of superstring theory.

A basic thermodynamic quantity one would like to compute for the thermal system corresponding to the background (3.1) is the free energy. It can be calculated as usual from the two terms in the action

$$\mathcal{I} = \mathcal{I}_{\rm grav} + \mathcal{I}_{\rm surf}\,. \tag{3.4}$$

The former is the Einstein-Hilbert action

$$\mathcal{I}_{\rm grav} = \frac{1}{2\kappa_{10}^2}\int_{\mathcal{M}} d^{10}x\sqrt{g}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}e^{-\phi}H_{(3)}^2\right)\,, \tag{3.5}$$

while the latter is the surface contribution [28]

$$\mathcal{I}_{\rm surf} = \frac{1}{\kappa_{10}^2}\oint_{\Sigma} K d\Sigma\,, \tag{3.6}$$

with $\mathcal{M}$ being a ten-volume enclosed by a nine-boundary Σ. The surface contribution is determined by the extrinsic curvature

$$K = \nabla_\mu n^\mu = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\, n^\mu\right)\,, \tag{3.7}$$

where n^μ is the boundary outward normal vector. It is chosen at a constant cutoff value of the radial coordinate r which eventually will be taken to infinity, see for instance [29].

To leading order, the free energy is just the classical action evaluated on the solution of the equations of motion of the gravity-matter system, divided by the inverse temperature β. In our case, the action (3.4) diverges as the radial cutoff is sent to infinity and hence needs to be renormalized. In the following we perform it in two different ways and check the consistency of the approaches.

3.1.1 Reference background renormalization

One way to renormalize the action (3.4) is to subtract the corresponding quantity for a reference background. This must be viewed as the equivalent of fixing the vacuum energy. The natural choice for the reference background is the extremal metric obtained from (3.1) by sending $z_0 \to 0$ and we shall proceed with it in the sequel.

We then evaluate expression (3.4) on the background with a radial cut-off $\mathcal{R}$, obtaining

$$\mathcal{I}_e = \frac{1}{2\kappa_{10}^2}\, V_5\, 2\pi^2\int_0^{\beta'} dt\left(-\int_0^{\mathcal{R}} dz\, z + \frac{9}{2}\mathcal{R}^2\right)\,, \tag{3.8}$$

for the extremal case and

$$\mathcal{I}_{ne} = \frac{1}{2\kappa_{10}^2} V_5 \, 2\pi^2 \int_0^\beta dt \left(- \int_{z_0}^{\mathcal{R}} dz \, z + \frac{1}{2} \left(9\mathcal{R}^2 - 5z_0^2 \right) \right) , \tag{3.9}$$

for the non-extremal one. In the expressions above, V_5 stands for the volume of the $\mathbf{R}^5$ along the branes, the factor $2\pi^2$ accounts for the volume of the $\mathbf{S}^3$ and β' and β are of course the periods of the compactified Euclidean time directions.

Notice that both bulk radial integrands are the same (just z). Thus, besides the surface contribution, differences arise from two sources, both due to the integration region: *i)* the radial coordinate for the non-extremal case is subject to $z \geq z_0$ while in the extremal case $z \geq 0$. *ii)* The periodicity β of the thermal circle in the non-extremal case is fixed to $\beta = 2\pi\sqrt{N}$; in the extremal solution β' is not fixed in the solution, but in order to make sense of the subtraction one must carefully adjust it in order that the lengths of the two thermal cycles at the cutoff coincide. This is done by setting $\beta' = \beta\sqrt{|g_{00}|}$, where g_{00} is evaluated at the cutoff.

The result is finally obtained by sending the cutoff to infinity [30]

$$\mathcal{I} = \lim_{\mathcal{R}\to\infty} [\mathcal{I}_{ne} - \mathcal{I}_e] = \lim_{\mathcal{R}\to\infty} \frac{(2\pi)^3}{2\kappa_{10}^2} \sqrt{N} \, V_5 \left[(2\mathcal{R}^2 - z_0^2) - 2\mathcal{R}\sqrt{\mathcal{R}^2 - z_0^2} \right] = 0 . \tag{3.10}$$

The free energy of the system vanishes.

3.1.2 Holographic renormalization

While the procedure just explained is successful, one is still restricted to use a reference background and may wonder whether the result depends on its choice. Moreover in some cases this reference background may not exist or just, as in the previous example, contains a singularity.

A second procedure overcomes these possible objections by using an effective approach [19, 6]. One makes use of the fact that the radial coordinate transverse to the branes is related to the energy scale in the dual field theory [31]. For example, the string propagating on the geometry (3.2) is related to a dual field theory, in the sense that gravity quantities at a given radius $\mathcal{R}$ correspond to field theory observables at a fixed energy (related to $\mathcal{R}$). The field theory observables, that live on the world volume of the NS5 branes, eventually must be rendered finite by renormalizing them. In order to do so in the gravity dual, one identifies the functional form of all possible sources of divergences that can be generically obtained on the world volume of the NS5. Writing down the metric (3.1) in the form

$$ds^2 = ds_{6+1}^2 + e^{2\sigma(z)} L^2 \, d\Omega_3^2 , \tag{3.11}$$

the prescription for the generic counterterm is [20]

$$\mathcal{I}_{ct} = \frac{A \, L^2 \, \Omega_3}{\kappa_{10}} \int_{\tilde{\mathcal{M}}} d^6x \, \sqrt{\tilde{g}_6} \, e^{3\sigma(z)} \, e^{B\phi(z)} , \tag{3.12}$$

where $\tilde{g}_6$ denotes the metric[3] induced on the brane world volume $\tilde{\mathcal{M}}$, computed at the cutoff $\mathcal{R}$. The term $e^{3\sigma(z)}$ comes from the coefficient of the three-sphere, of volume Ω_3, in the determinant of the full metric. The constants A, B are then tuned in order to cancel the divergences in the action $\mathcal{I}_{ne}$ (3.9). In our case the choice $A = 2$, $B = -\frac{1}{4}$ is uniquely selected by the requirement of absence of divergences in the $\mathcal{R} \to \infty$ limit, and leads to the same result as before

$$\mathcal{I} = \lim_{\mathcal{R}\to\infty} (\mathcal{I}_{ne} - \mathcal{I}_{ct}) = 0\,. \tag{3.13}$$

3.1.3 Energy and entropy

Since the free energy of the system, $F = \mathcal{I}/\beta$, is vanishing, from the usual thermodynamic relation $F = E - TS$ one expects that the energy is proportional to the entropy.

For a stationary spacetime admitting foliations by spacelike hypersurfaces Σ_t the conserved energy enclosed in a shell is given by the ADM relation [32, 33] (the Newton constant $16\pi G_N = 2\kappa_{10}^2$ is explicitated for latter convenience)

$$E = -\frac{1}{8\pi G_{10}} \oint_{\Sigma_t} (K - K_0)\sqrt{|g_{00}|}\, d\Sigma_t\,, \tag{3.14}$$

where now the non-extremal K (extremal K_0) extrinsic curvature is obtained at a fixed time slice. One can compute the energy density for the LST background (3.1) by using the definition (3.7), obtaining

$$e \equiv \frac{E}{V_5} = \frac{\pi\, z_0^2}{4\, G_{10}}\,. \tag{3.15}$$

Furthermore, the entropy density calculated from the area of the black hole horizon with the Bekenstein-Hawking relation

$$S = \frac{\text{Area}}{4G_{10}} \tag{3.16}$$

gives

$$s \equiv \frac{S}{V_5} = \frac{\pi^2\sqrt{N}\, z_0^2}{2\, G_{10}}\,. \tag{3.17}$$

Thus, using the fact that $T = \frac{1}{2\pi\sqrt{N}}$, it follows that the LST background satisfies the usual thermodynamic relations and $E = TS$.

3.2 Thermodynamics of $\mathcal{N} = 1$ SQCD-like theories

In the previous section we have reviewed how the reference background and holographic renormalization work in the simple case of LST, where one does not have any flavor branes. We now turn to the more involved case of the flavor-backreacted metric (2.2), dual to SQCD-like theories with $N_f = 2N_c$ flavors. In order to study its thermodynamics, we follow the same steps as in the LST model, but including now the DBI term for the flavor branes. We shall find a completely consistent picture by using the straightforward extensions of the usual procedures.

[3] Note that, having an undetermined power of the dilaton, there is no difference between the Einstein and the string frame.

As a first thing, we compute the free energy from the Euclidean action. The latter reads in this case

$$\mathcal{I} = \mathcal{I}_{\text{grav}} + \mathcal{I}_{\text{flavor}} + \mathcal{I}_{\text{surf}} \,, \tag{3.18}$$

where the first and third terms are the standard ones,

$$\mathcal{I}_{\text{grav}} = \frac{1}{2\kappa_{10}^2}\int_{\mathcal{M}} d^{10}x\sqrt{-g}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}e^{\phi}F_{(3)}^2\right), \qquad \mathcal{I}_{\text{surf}} = \frac{1}{\kappa_{10}^2}\oint_{\Sigma} Kd\Sigma \,, \tag{3.19}$$

where $K = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\,n^\mu\right)$. $\mathcal{I}_{\text{flavor}}$ is instead given by the inclusion of fundamental matter [18] through a family of N_f D5-branes originally extended along the $t, x_1, x_2, x_3, r, \psi$ directions and finally evenly smeared in the extra compact directions $\theta, \varphi, \bar{\theta}, \bar{\varphi}$. It is given by

$$\mathcal{I}_{\text{flavor}} = -\frac{T_5 N_f}{(4\pi)^2}\int_{\mathcal{M}} d^{10}x \sin\theta\sin\bar{\theta}e^{\phi/2}\sqrt{-\tilde{g}_6} + \frac{T_5 N_f}{(4\pi)^2}\int_{\mathcal{M}} \text{Vol}(\mathcal{Y}_4)\wedge C_{(6)} \,, \tag{3.20}$$

where in string units $T_5 = 1/(2\pi)^5$.

3.2.1 Free energy from the reference background renormalization

As in the LST model the temperature of our background (2.2) is fixed to $T = \frac{1}{2\pi\sqrt{N_c}}$ and so it does not depend on the mass, or size, of the black hole. We expect in such a case, as we know from the study of the LST, an energy proportional to the entropy and therefore a vanishing free energy. We shall show that this is indeed the case, by using as reference background the zero temperature one (2.1) in order to renormalize the infinite free energy of the thermal background (2.2).

Let us first consider the "bulk" contributions to the action $\mathcal{I}$

$$\mathcal{I}_{\text{bulk}} \equiv \mathcal{I}_{\text{grav}} + \mathcal{I}_{\text{flavor}} \,. \tag{3.21}$$

As a first thing, one notices that the specific structure of the three-form (2.4) guarantees that the Chern-Simons term in (3.20) vanishes (both in the zero temperature and in the finite temperature cases). It is then a matter of straightforward computation to evaluate the remaining parts of $\mathcal{I}_{\text{bulk}}$ on the background. In the case of the finite temperature solution (2.2) one gets, for the integrand in $\mathcal{I}_{\text{bulk}}$,

$$\text{Integrand}_{\text{bulk}} = \frac{e^{2(r+\phi_0)}N_c^2\sin\theta\sin\bar{\theta}}{(4-\xi)\xi} \,, \tag{3.22}$$

which is independent of r_0. As a consequence, the same integrand will be obtained for the zero temperature solution (2.1). This is a very typical situation that we have already seen in the LST and it also happens in the case of the AdS black hole [3]. This means that the effects of the background subtraction, as regards the bulk contributions, are only due to differences in the integration region. There are two such differences. In the finite temperature case the radial integration goes from r_0 to the cutoff $\mathcal{R}$ whereas in the zero temperature case it starts at $r = -\infty$. The second difference is the range of integration of the Euclidean time variable. In the zero temperature case this range, β', is arbitrary and must be adjusted to the value $\beta' = \beta\sqrt{1 - e^{2(r_0 - \mathcal{R})}}$, where $\beta = 2\pi\sqrt{N_c}$, such that the geometry of this background coincides with that of the finite temperature one at the cutoff $\mathcal{R}$. With these ingredients one can compute the finite temperature bulk contribution, subtract from it the zero temperature contribution at the cutoff $\mathcal{R}$ and finally send $\mathcal{R}$ to infinity, getting

$$\frac{\mathcal{I}_{\text{bulk}}}{V_3} = \frac{(2\pi)^4 e^{2(r+\phi_0)} N_c^{\frac{5}{2}}}{2\kappa_{10}^2 (4-\xi)\xi}\,, \tag{3.23}$$

with $V_3 = \text{Vol}(\mathbf{R}^3)$.

Next we move to the boundary contribution $\mathcal{I}_{\text{surf}}$. Using again β' and β for the thermal cycles and performing the subtraction of the contributions from the two backgrounds we get, after sending $\mathcal{R}$ to infinity,

$$\frac{\mathcal{I}_{\text{surf}}}{V_3} = -\frac{(2\pi)^4 e^{2(r+\phi_0)} N_c^{\frac{5}{2}}}{2\kappa_{10}^2 (4-\xi)\xi}\,. \tag{3.24}$$

Thus the free energy density vanishes

$$f = \frac{1}{\beta V_3}(\mathcal{I}_{\text{bulk}} + \mathcal{I}_{\text{surf}}) = 0\,, \tag{3.25}$$

as expected. Of course, the novel ingredient, that is the DBI term for the flavors $\mathcal{I}_{\text{flavor}}$, is crucial in order to obtain the exact coefficient in (3.23) such that the free energy is zero.

In a sense, even if the result is the expected one, our computation might look somewhat suspicions, or problematic, for the zero temperature background has a singularity at $r \to -\infty$. Although being classified as a "good" singularity, it signals the breakdown of the supergravity approximation, and in view of that one may wonder about the legitimacy of the result. So, in order to test the solidity of our result we shall do two independent checks. On one hand, we can compute separately the energy and the entropy associated with our finite temperature background and check whether the free energy vanishes. This will be done later. On the other hand we can use a holographic counterterm subtraction instead of the background subtraction. We devote the next subsection to this last issue.

3.2.2 Free energy from holographic renormalization

We are now going to renormalize the action (3.18) using the holographic renormalization prescription. While for the gravitational part this is fairly standard and we have reviewed it in the previous case of LST, for the flavor part its implementation is still not a fully tested procedure. Namely, while it has been studied for the renormalization of brane actions on curved backgrounds, as far as we know it has never been used for the case where the brane action is a source of the gravitational background. Thus, here we provide the first full-fledge example of such procedure, where we confirm that the prescriptions in [6, 19, 20] give consistent and unique results.

As in the previous section, it is a matter of straightforward computation to plug the finite temperature solution (2.2) in the action (3.18) and integrate up to a radial cutoff $\mathcal{R}$.

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The result for the usual gravitational part (3.19), $\mathcal{I}_{\rm gravity} \equiv \mathcal{I}_{\rm grav} + \mathcal{I}_{\rm surf}$, reads

$$\mathcal{I}_{\rm gravity} = \frac{e^{2\phi_0} N_c^{\frac{5}{2}} V_3}{\pi^3} \cdot \frac{e^{2\mathcal{R}}[16 + (4-\xi)\xi] - e^{2r_0}[8 + (4-\xi)\xi]}{8(4-\xi)\xi}\,, \tag{3.26}$$

while the contribution of the flavor branes (3.20) is

$$\mathcal{I}_{\rm flavor} = \frac{e^{2\phi_0} N_c^{\frac{5}{2}} V_3}{\pi^3} \cdot \frac{(e^{2r_0} - e^{2\mathcal{R}})}{8}\,. \tag{3.27}$$

Each of the two pieces must be renormalized.

For what concerns the gravitational part $\mathcal{I}_{\rm gravity}$ (3.26), as we saw in the LST case the general holographic renormalization setup concerns a background generated by a stack of Dp-branes with a sphere transverse to the branes, which can be dimensionally reduced (see appendix E of [20]). There are some notable differences with our present case. In our D5-brane case the transverse space has a more complicated structure, a fibered S^3, and, in addition, the D5-branes wrap a two-cycle. These differences notwithstanding, we just take a slight generalization of the proposal in [20] in the sense that there should exist an effective counterterm which, when written before undertaking any dimensional reduction of the transverse space, must be of the form

$$\mathcal{I}_{\rm ctgravity} = \frac{A}{L\kappa_{10}^2} \int_{\partial\mathcal{M}} d^9x \sqrt{h}\, e^{B\phi}\,, \tag{3.28}$$

where $L = \sqrt{N_c}$ is the scale factor in front of the compact part of the metric (see (3.11) for comparison), $\partial\mathcal{M}$ is the boundary of the manifold at $r = \mathcal{R}$, h is the boundary metric, and A encompasses a global constant which, together with the constant B, is determined in order to cancel the divergences in (3.26). Cancellation of the large $\mathcal{R}$ divergence in $\mathcal{I}_{\rm gravity} + \mathcal{I}_{\rm ctgravity}$ then fixes $A = -[16 + (4-\xi)\xi]/8$ and $B = -1/4$. Thus

$$\mathcal{I}_{\rm gren} \equiv \lim_{\mathcal{R}\to\infty} (\mathcal{I}_{\rm gravity} + \mathcal{I}_{\rm ctgravity}) = -\frac{e^{2(r_0+\phi_0)} N_c^{5/2} V_3}{16\pi^3}\,. \tag{3.29}$$

Analogously, for the flavor part $\mathcal{I}_{\rm flavor}$ (3.27) we use the prescription in [20], taking the counterterm

$$\mathcal{I}_{\rm ctflavor} = 2\pi T_5 N_f L^2 A' \int d^4x \sqrt{\gamma} e^{2\sigma'} e^{B'\phi}\,, \tag{3.30}$$

where γ is the induced metric on the 4d part of the world-volume, $2\pi e^{2\sigma'} = 2\pi e^{(r+\phi_0)/4}/2$ comes from the wrapped ψ-direction, and A', B' must be adjusted in order to cancel the divergence in $\mathcal{I}_{\rm flavor} + \mathcal{I}_{\rm ctflavor}$. The latter requirement then fixes $A' = -1$ and $B' = 3/4$. In the usual approach to holographic renormalization, the counterterm above corresponds to a volume factor. In general, there could be other counterterms proportional to the curvature of the induced metric at the radial cut-off [6]. These terms vanish in our case, as the flavor brane embedding is such that the induced metric is Ricci-flat (it is the metric for flat space times a circle).

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The renormalized flavor contribution is thus

$$\mathcal{I}_{\rm fren} \equiv \lim_{\mathcal{R}\to\infty}(\mathcal{I}_{\rm flavor} + \mathcal{I}_{\rm ctflavor}) = \frac{e^{2(r_0+\phi_0)} N_c^{5/2} V_3}{16\pi^3}\,, \tag{3.31}$$

and so the free energy density is vanishing

$$f = \frac{1}{\beta V_3}(\mathcal{I}_{\rm gren} + \mathcal{I}_{\rm fren}) = 0\,. \tag{3.32}$$

This agrees with the result previously obtained by using the background subtraction method, thus putting the whole procedure on a solid ground.

Let us note that we could have joined the two separate pieces, the standard gravitational one and flavor one, in a unique term, for which we could have used just one counterterm. The latter would have of course produced again a vanishing free energy. Nevertheless, since this procedure would have hidden the structure of the counterterms, above we choose to present the separate renormalization of the two pieces.

3.2.3 Energy and entropy

A direct procedure to check the vanishing of the free energy is to compute independently the energy and the entropy. We use the relation (3.14) to compute the conserved ADM energy. The energy density turns out to be

$$e = \frac{1}{8\pi G_N}\frac{e^{2(r_0+\phi_0)} N_c^2 (4\pi)^3}{2(4-\xi)\xi} = \frac{8\lambda^4}{(4-\xi)\xi} T^4\,, \tag{3.33}$$

where we defined $\lambda \equiv e^{r_0+\phi_0} N_c$, which is the quantity that must be fixed and large in order for the gravitational description of the system (2.2) to be reliable. In addition we used the fact that in string units $\alpha' = 1$ and in the Einstein frame the Newton constant is given by $G_N = \kappa_{10}^2/8\pi = 2^3\pi^6/e^{2(r_0+\phi_0)}$.

As for the entropy density, as a quarter of the area of the horizon of the metric (2.2) at $r = r_0$, it is just [17]

$$s = \frac{1}{4G_N}\frac{e^{2(r_0+\phi_0)} N_c^{\frac{5}{2}} (4\pi)^3}{2(4-\xi)\xi} = \frac{8\lambda^4}{(4-\xi)\xi} T^3\,. \tag{3.34}$$

Thus the free energy density $f = e - Ts$ vanishes, as expected.

Note the nice feature that the energy and entropy density are invariant under the change $\xi \to 4-\xi$, which is supposed to correspond to a Seiberg duality for the field theory dual to the zero energy solution [18].[4] This is of course a trivial consequence of the invariance of the background under this transformation (together with $(\theta,\phi) \leftrightarrow (\bar\theta,\bar\phi)$).

All in all, the thermodynamics of this model is very similar to the LST one, the temperature is fixed irrespectively of the horizon size and the free energy vanishes identically. The thermal system is described by a black hole at the fixed Hagedorn temperature. In LST it was argued, by studying the first string corrections to this situation or by studying

[4]We thank C. Nunez for this comment.

compactified systems, that there are thermal instabilities, as the specific heat turns out to be negative [34 – 36, 29, 37]. Given the similarities of the two models, one can expect similar instabilities in our thermal background (2.2) too, but a much more involved analysis, which is outside the scope of this paper, would be required in order to check this statement.

4. Jet quenching parameter

The solution (2.2) we are considering in this note is the only known ten dimensional black hole dual to a 4d finite temperature field theory that includes the backreaction of many flavor degrees of freedom. This is a sufficient motivation for the investigation of its properties, despite its thermodynamics resembles the Little String Theory one and may eventually reveal instabilities. In fact, as stressed in the Introduction, there has been evidence that string theory may provide some insight into the study of properties of the Quark Gluon Plasma. Here we are interested in the effects of the backreaction of fundamental flavors on one specific plasma observable, namely the jet quenching parameter.

In [17] the authors calculated for the background (2.2) some quantities relevant to the explanation of the huge energy loss of colored probes in the QGP, namely the jet quenching parameter [9] and the relaxation time [10, 11]. While the result for the latter turns out to be quite similar to the ones in previous "unflavored" string models, the jet quenching parameter is unexpectedly vanishing. This is not an effect of the presence of fundamental flavors, since it is a common features to all the theories coming from D5-branes [17]. But, the relevant observation for our scope is that in order to perform the computation, a specific string configuration was used, introduced in [9] for the $AdS_5 \times S^5$ black hole dual to $\mathcal{N} = 4$ SYM. The corresponding string extends from infinity up to the horizon of the black hole. In [22] it was observed that apparently that configuration is not the minimal energy one. As such, the path integral is dominated by other configurations and accordingly the jet quenching parameter should be calculated using the latter.

Thus, considering this possibility, in this section we perform a careful analysis of the string configurations that could be used to calculate the jet quenching parameter, as proposed in [22]. We will consider the set-up in which we start from strings at velocity $V \neq 1$ and finite mass m of the dual quarks, with space-like world-sheet, and take the limit of $V \to 1$ and infinite m at the end, thus giving to the final result a physical origin. As we will see, though, among all the possibilities, the only configuration that seems to give a sensible physical result is the one used in [17], that gives zero jet quenching parameter.

To be concrete, the jet quenching parameter is calculated from string theory as the coefficient of the $L^- L^2$ term in the action for a macroscopic string, spanning a Wilson loop with a light-like dimension L^- much larger that the spatial one, of size L [9]. It corresponds to the infinite mass, infinite velocity limit of a string describing a quark-antiquark pair. The latter provides a good measure of the jet quenching parameter when it has a space-like world-sheet [16].

Thus, consider a test string, in the static gauge, representing a dipole moving with velocity v along the x_1 direction, perpendicularly to its extension along x_2. We will work in the string frame and with the radial coordinate $z = e^{\frac{r}{2}}$, such that the metric (2.2) reads

$$ds_T^2 = e^{\phi_0} z^2 \Bigg[-\mathcal{F} dt^2 + d\vec{x}_3^2 + N_c \bigg(\frac{4}{z^2 \mathcal{F}} dz^2 + \frac{1}{\xi}(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.1}$$
$$+ \frac{1}{4-\xi}(d\bar{\theta}^2 + \sin^2\bar{\theta} d\bar{\varphi}^2) + \frac{1}{4}(d\psi + \cos\theta d\varphi + \cos\bar{\theta} d\bar{\varphi})^2 \bigg) \Bigg], \qquad \mathcal{F} = 1 - \frac{z_0^4}{z^4},$$

and of course z_0 is the position of the horizon. The relevant string configuration is then given by

$$t = \tau, \qquad x_1 = v\tau, \qquad x_2 = \sigma, \qquad z(\sigma). \tag{4.2}$$

The string is attached to a probe "flavor" D5-brane, whose radial position z_b is dual to the quark mass m. The separation of the quark-antiquark pair is denoted by L. The only stable configuration is that which keeps the plane where the string lies perpendicular to the velocity.

As pointed out in [22], and can be deduced from (2.2), v is not the proper dipole velocity. The true velocity of the dipole is $V = v/\sqrt{1 - z_0^4/z_b^4}$. We are interested in spacelike configurations on which the lightlike limit $V \to 1$ will be taken. This means that $v \to 1$ and $z_b \to \infty$.[5] It will make a difference whether the limit for v is taken form below $v \to 1^-$ of from above $v \to 1^+$. The spacelike condition $G_{\tau\tau} \equiv g_{\mu\nu}\partial_\tau x^\mu \partial_\tau x^\nu > 0$ will make the induced action on the world-sheet, $S = \frac{1}{2\pi}\int d\tau d\sigma \sqrt{-G}$, imaginary. Therefore e^{iS} will be a real exponential. Following [22] we will take the minus sign for the exponent $e^{iS} = e^{-S_r}$. Henceforth we will work directly with S_r.

4.1 Spacelike configurations with $v \to 1^-$

The induced action on the world-sheet is

$$S_r = \frac{1}{2\pi}\int d\tau d\sigma \sqrt{G} = \frac{1}{2\pi}\int d\tau d\sigma \frac{e^{\phi_0}}{\gamma}\sqrt{(z_0^4\gamma^2 - z^4)\left(1 + 4N_c z^2 z'^2 \frac{1}{z^4 - z_0^4}\right)}, \tag{4.3}$$

where we have introduced the standard notation $\gamma^2 = 1/(1 - v^2)$.

Let us first analyze the possible location of the turning point for the string configuration. The σ-independent conserved quantity of (4.3) is

$$\frac{z_0^4\gamma^4 - z^4}{\frac{4N_c z^2 z'^2}{z^4 - z_0^4} + 1} \equiv h > 0. \tag{4.4}$$

Thus,

$$z'^2 = \left(\frac{z_0^4\gamma^2 - z^4 - h}{h}\right)\left(\frac{z^4 - z_0^4}{4N_c z^2}\right). \tag{4.5}$$

Since $z^4 \geq z_0^4$ for the string configuration, obviously one needs $z^4 \leq z_0^4\gamma^2 - h$ to guarantee $z'^2 \geq 0$. Defining z_t by $z_t^4 \equiv z_0^4\gamma^2 - h \geq z^4 \geq z_0^4$, one has

$$z'^2 = \left(\frac{z_t^4 - z^4}{z_0^4\gamma^2 - z_t^4}\right)\left(\frac{z^4 - z_0^4}{4N_c z^2}\right), \tag{4.6}$$

[5] For $v < 1$ there is in principle the finite case when $z_b^4 = \frac{z_0^4}{1-v^2}$, but this case will be discarded in due time.

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with $z_0 \le z \le z_b \le z_t \le z_0\sqrt{\gamma}$, where z_b is the location of the brane.

The possible turning points, for which $z'^2 = 0$, can only be z_0 or z_t. Let us examine both cases.

4.1.1 The *down* configuration

First consider the case when the string configuration reaches the horizon at z_0, which will be the turning point. Using variables $z = yz_0$, the distance between the two quarks at radial coordinate z_b is

$$\frac{L}{\beta} = \frac{1}{\beta}\int d\sigma = \frac{2}{\beta}\int_1^{y_b}\frac{dy}{y'} = \frac{2}{\pi}\alpha\gamma\int_1^{y_b} dy\frac{y}{\sqrt{(y_t^4-y^4)(y^4-1)}}\,, \tag{4.7}$$

where β is the inverse temperature and the definition $\alpha^2 = 1 - y_t^4/\gamma^2 \le 1$ has been used. The action can be written as

$$S_r = \frac{1}{2\pi}\frac{e^{\phi_0}}{\gamma}4\sqrt{N_c}\,\hat{t}\,z_0^2\int_1^{y_b} dy\frac{y(\gamma^2-y^4)}{\sqrt{(y_t^4-y^4)(y^4-1)}}\,, \tag{4.8}$$

where $\hat{t}$ is the time interval the dipole propagates.

In the limit $\gamma \to \infty$, while keeping y_b finite, one must also send $y_t \to \infty$ in order to keep L finite. In this limit, L becomes

$$\frac{L}{\beta} \approx \frac{2}{\pi}\frac{\alpha}{\sqrt{1-\alpha^2}}\int_1^{y_b} dy\frac{y}{\sqrt{y^4-1}} = \frac{\alpha}{\pi\sqrt{1-\alpha^2}}\,\mathrm{arccosh}\,(y_b^2)\,. \tag{4.9}$$

Notice that in a next step we will send $y_b \to \infty$ and then α must vanish in order to keep L finite. In the same limit, S_r becomes

$$S_r \approx \frac{1}{2\pi}\frac{e^{\phi_0}}{\sqrt{1-\alpha^2}}4\sqrt{N_c}\,\hat{t}\,z_0^2\int_1^{y_b} dy\frac{y}{\sqrt{y^4-1}} = \frac{2\pi\lambda\,\hat{t}}{\beta^2}\frac{L}{\alpha}\,, \tag{4.10}$$

where $\lambda \equiv e^{\Phi_0}\,z_0^2\,N_c$. To this action we must subtract the infinite mass quark contribution associated with the action S_0 for two straight strings stretching from the brane to the horizon

$$S_0 = \frac{1}{2\pi}\frac{e^{\phi_0}}{\gamma}4\sqrt{N_c}\,\hat{t}\,z_0^2\int_1^{y_b} dy\,y\sqrt{\frac{\gamma^2-y^4}{y^4-1}} \approx \frac{2\pi\lambda\,\hat{t}}{\beta^2}\sqrt{1-\alpha^2}\,\frac{L}{\alpha}\,. \tag{4.11}$$

In the required limit $\alpha \to 0$, the subtraction gives a vanishing renormalized action

$$S_{\text{ren}} = \lim_{\alpha\to 0}(S_r - S_0) = 0\,. \tag{4.12}$$

This result also holds if both limits, $\gamma \to \infty$ and $y_b \to \infty$ are taken simultaneously — with $y_b \le \sqrt{\gamma}$. Thus, being the coefficient of this renormalized action identified in [9] with the jet quenching parameter $\hat{q}$, the conclusion is that for this configuration $\hat{q} = 0$, in accordance with what argued in [17]. The main concern is if this result can be altered by possible different configurations.

4.1.2 The *up* configuration

Now consider the turning point at $y_t \geq y_b$. With the approximation $y_b \gg 1$, which is always legitimate because at the end we send $y_b \to \infty$, the distance between the quarks is

$$\frac{L}{\beta} \approx \frac{2}{\pi} \alpha\gamma \int_{y_b}^{y_t} \frac{dy}{y\sqrt{(y_t^4 - y^4)}}\,, \tag{4.13}$$

and the action of the configuration becomes

$$S_r \approx \frac{4\,\lambda\,\hat{t}}{\beta\gamma} \int_{y_b}^{y_t} \frac{(\gamma^2 - y^4)\;dy}{y\;\sqrt{(y_t^4 - y^4)}}\,. \tag{4.14}$$

Now we have

$$1 < y_b \leq y \leq y_t \leq \sqrt{\gamma}.$$

Let us define $\eta = \frac{y_t}{y_b} \geq 1$. Then the parameters (y_b, y_t, γ) can be traded for (y_b, α, η). The distance L becomes

$$\frac{L}{\beta} \approx \frac{\alpha}{\pi} \frac{1}{\sqrt{1-\alpha^2}} \operatorname{arccosh}(\eta^2)\,, \tag{4.15}$$

out of which we determine

$$\eta(L, \alpha) = \left(\cosh(\frac{\pi\,\sqrt{1-\alpha^2}\,L}{\beta\alpha})\right)^{\frac{1}{2}}. \tag{4.16}$$

S_r is then written as

$$\begin{aligned} S_r &\approx \frac{2\lambda\,\hat{t}}{\beta}\left(-\sqrt{1-\alpha^2}\sqrt{1-\frac{1}{\eta^4}} + \frac{\operatorname{arccosh}(\eta^2)}{\sqrt{1-\alpha^2}}\right) \\ &= \frac{2\lambda\,\hat{t}}{\beta}\left(-\sqrt{1-\alpha^2}\tanh(\frac{\pi\,\sqrt{1-\alpha^2}\,L}{\beta\alpha}) + \frac{\pi}{\beta}\frac{L}{\alpha}\right). \end{aligned} \tag{4.17}$$

Notice, from the definitions of η and α, the relation

$$\frac{y_b^4}{\gamma^2} = \frac{1-\alpha^2}{\eta^4} = \frac{1-\alpha^2}{\left(\cosh(\frac{\pi\,\sqrt{1-\alpha^2}\,L}{\beta\alpha})\right)^2}\,, \tag{4.18}$$

that will prove useful in the following.

Our natural free parameters are γ, L, y_b, with the restriction $y_b^2 \leq \gamma$. The values of α and η are then read off from (4.18) and (4.16). Let us examine the possible limits leading to $V \to 1$. One can easily discard the case with $v < 1$ and finite y_b. This setting implies $y_b^2 = \gamma$, but, according to (4.18), this is to set $1-\alpha^2 = \left(\cosh(\frac{\pi\,\sqrt{1-\alpha^2}\,L}{\beta\alpha})\right)^2$ which is clearly incompatible.

Our task now is to take both limits $v \to 1^-$ (that is, $\gamma \to \infty$) and $y_b \to \infty$ subject to the restriction $y_b \leq \sqrt{\gamma}$.

(a) Let us then first take the limit $\gamma \to \infty$. Inspection of (4.18) shows that there are two ways to do it, while keeping y_b finite: either by sending $\alpha \to 1$, which implies $\eta \to 1$, or by sending $\alpha \to 0$, which implies $\eta \to \infty$. Let us examine more closely both cases, always keeping y_b finite.

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(a.1) If $\alpha \to 1$, the consequence $\eta \to 1$ dictates $y_t \to y_b$, that is, the string configuration does not actually leave the brane. The action written above remains finite in this limit, $S_r = \frac{2\pi\lambda\hat{t}}{\beta^2} L$. This result is independent of the value y_b. The standard procedure of subtracting from S_r the action S_0 corresponding to two straight lines corresponding to the two bare quark masses from y_b to the horizon, will yield a negative infinite value for the renormalized action when y_b is sent to infinity, since $S_0 \sim \operatorname{arccosh}(y_b^2)$.

(a.2) If we take the opposite limit, $\alpha \to 0$, then $\eta \to \infty$ and therefore the turning point goes to infinity, $y_t \to \infty$. Now the action is infinite and some subtraction is mandatory to obtain a finite result, but again it seems not to exists any finite subtraction because the infinite piece in S_r goes as $\frac{L}{\alpha}$ whereas the action for the two straight lines goes as $\ln(y_b)$.[6] Since we have to take first the $\alpha \to 0$ limit at fixed y_b, we find a positive infinite result in this case.

(b) The remaining case to explore is to take both limits, $\gamma \to \infty$, $y_b \to \infty$, simultaneously such that (4.18) does not go to zero. So we fix α with $0 < \alpha < 1$. In such case we run into the same problem we met before. Now S_r is finite before subtraction, but the subtraction piece is an infinite quantity when $y_b \to \infty$, and so we end up again with a (negative) infinite value for the renormalized action.

In conclusion, none of the "*up*" configurations with $v < 1$ allows for a lightlike limit with a finite renormalized action. The configurations with a negative infinite value for the action would dominate the path integral. Nevertheless, since the subtraction piece just gets rid of the infinite mass of the quarks in the limit $y_b \to \infty$, the fact that, for these configurations, S_r is finite before subtraction means that the infinite quark masses are already canceled by an infinite binding energy. Such configurations, which have infinite action, have hardly a physical meaning in the dual field theory, so we discard this possibility.

4.2 Spacelike configuration with $v \to 1^+$

We introduce the parameter $\tilde{\gamma}^2 = \frac{1}{v^2-1}$. Let us first analyze the possible location of the turning point for the string configuration. The σ-independent conserved quantity of the action is

$$\frac{z_0^4\tilde{\gamma}^4 + z^4}{\frac{4\,N_c\,z^2(z')^2}{z^4-z_0^4} + 1} \equiv h > 0\,. \tag{4.19}$$

Thus,

$$(z')^2 = \left(\frac{z^4 + z_0^4\tilde{\gamma}^2 - h}{h}\right)\left(\frac{z^4 - z_0^4}{4\,N_c\,z^2}\right). \tag{4.20}$$

If $h < z_0^4 + z_0^4\tilde{\gamma}^2$, the only possible turning point is at z_0, because $z \geq z_0$. The critical case $h = z_0^4 + z_0^4\tilde{\gamma}^2$ makes z_0 to be a simple zero of z'; this has the consequence that the distance between the two quarks can not be kept finite, and thus we discard this

[6]Trying to adjust both limits, $\alpha \to 0$ and $y_b \to \infty$, in order to obtain a finite renormalized action is an artificial procedure which lacks of any reasonable support.

configuration. Finally, for $h > z_0^4 + z_0^4\tilde{\gamma}^2$ one can define $z_t^4 \equiv h - z_0^4\tilde{\gamma}^2 > z_0^4$, and we can write in accordance

$$(z')^2 = \left(\frac{z^4 - z_t^4}{z_0^4\tilde{\gamma}^2 + z_t^4}\right)\left(\frac{z^4 - z_0^4}{4\,N_c\,z^2}\right), \tag{4.21}$$

with the only turning point in z_t because if it were z_0, there will always exist at least some region where $(z')^2$ becomes negative, no matter where is the location z_b of the brane (remind that $z_0 < z_t$).

4.2.1 The *down* configuration

Let us first consider the case when the string configuration reaches the horizon at z_0, which will be the turning point. Using as variable $z = yz_0$, the distance L between the two quarks at radial coordinate z_b is given by

$$\frac{L}{\beta} = \frac{1}{\beta}\int d\sigma = \frac{2}{\beta}\int_1^{y_b}\frac{dy}{y'} = \frac{2}{\pi}\int_1^{y_b}\frac{1}{\sqrt{y^4/\tilde{h} + \tilde{\gamma}^2/\tilde{h} - 1}}\frac{y}{\sqrt{y^4 - 1}}, \tag{4.22}$$

with $\tilde{h} = h/z_0^4$. In the same variables the action can be written as

$$S_r = \frac{1}{2\pi}\frac{e^{\phi_0}\,\hat{t}}{\tilde{\gamma}}\,4\sqrt{N_c}z_0^2\int_1^{y_b}\frac{y\,(\tilde{\gamma}^2 + y^4)}{\sqrt{(y^4 + \tilde{\gamma}^2 - \tilde{h})(y^4 - 1)}}\,. \tag{4.23}$$

We will deal essentially with two possible different limits.

(a) In the limit $\tilde{\gamma} \to \infty$, while keeping y_b finite, one must also send $\tilde{h} \to \infty$ in order to keep L finite. Indeed one must have a finite quotient, $\lim(\tilde{\gamma}^2/\tilde{h}) \equiv c > 1$. In this limit, L becomes

$$\frac{L}{\beta} \approx \frac{2}{\pi}\frac{1}{\sqrt{c-1}}\int_1^{y_b}\frac{y}{\sqrt{y^4 - 1}} = \frac{1}{\pi\sqrt{c-1}}\,\mathrm{arccosh}(y_b^2)\,, \tag{4.24}$$

and S_r,

$$S_r \approx \frac{1}{2\pi}4\sqrt{N_c}\hat{t}e^{\phi_0}z_0^2\sqrt{\frac{c}{c-1}}\,\frac{1}{2}\,\mathrm{arccosh}(y_b^2) = \frac{2\pi\lambda\,\hat{t}}{\beta^2}\sqrt{c}\,L\,. \tag{4.25}$$

When practicing the next limit, $y_b \to \infty$, c must go to infinity too in order to keep L finite. The action S_r becomes infinite and is renormalized by subtracting the infinite mass quark contribution associated with the action S_0. Before taking $y_b \to \infty$, S_0 is given by

$$S_0 = \frac{1}{2\pi}4\sqrt{N_c}\hat{t}e^{\phi_0}z_0^2\,\frac{1}{2}\,\mathrm{arccosh}(y_b^2) = \frac{2\pi\lambda\,\hat{t}}{\beta^2}\sqrt{c-1}\,L\,. \tag{4.26}$$

Therefore, again the renormalized action vanishes for the "*down*" configuration,

$$S_{\mathrm{ren}} = \lim_{c\to\infty}(S_r - S_0) = 0\,, \tag{4.27}$$

and accordingly the jet quenching parameter is zero.

(b) As a second possibility we analyze the case where we first take the $y_b \to \infty$ limit before taking $\tilde{\gamma} \to \infty$. Notice that

$$
\begin{aligned}
S_r - S_0 &= \frac{4\lambda\,\hat{t}}{\beta}\,\frac{1}{\tilde{\gamma}} \int_1^\infty y\sqrt{\frac{\tilde{\gamma}^2 + y^4}{y^4 - 1}} \left(\sqrt{\frac{\tilde{\gamma}^2 + y^4}{y^4 + \tilde{\gamma}^2 - \tilde{h}}} - 1 \right) \\
&\approx \frac{2\lambda\,\hat{t}}{\beta}\,\frac{\tilde{h}}{\tilde{\gamma}} \int_1^\infty \frac{y}{\sqrt{(\tilde{\gamma}^2 + y^4)(y^4 - 1)}}\,,
\end{aligned} \tag{4.28}
$$

with $\tilde{h} < \tilde{\gamma}^2 + 1$. Using a similar expansion for L, to first order in $\tilde{h}/\tilde{\gamma}^2$

$$
\frac{L}{\beta} \approx \frac{2}{\pi}\sqrt{\tilde{h}} \int_1^\infty \frac{y}{\sqrt{(y^4 + \tilde{\gamma}^2)(y^4 - 1)}}\,, \tag{4.29}
$$

which shows that, at this leading order,

$$
S_r - S_0 \approx \frac{4\lambda\,\hat{t}}{\beta}\,\frac{1}{2}\,\frac{\pi}{2}\,\frac{\sqrt{\tilde{h}}}{\tilde{\gamma}}\,\frac{L}{\beta} = \frac{\pi\,\lambda\,\hat{t}}{\beta^2}\,\frac{1}{\sqrt{c}}\,L\,, \tag{4.30}
$$

which again vanishes when $c \to \infty$ (the limit $c \to \infty$ must be taken when $\tilde{\gamma} \to \infty$ in order to keep L finite).

4.2.2 The *inner* configuration

In this case we will make use of the parameter $\tilde{\alpha}^2 = 1 + \frac{y_t^4}{\tilde{\gamma}^2}$. The distance between the two quarks is

$$
\frac{L}{\beta} = \frac{2}{\beta} \int_{z_t}^{z_b} \frac{dz}{z'} = \frac{2}{\pi}\,\tilde{\alpha}\tilde{\gamma} \int_{y_t}^{y_b} \frac{y dy}{\sqrt{(y^4 - y_t^4)(y^4 - 1)}}\,. \tag{4.31}
$$

Notice that, as said above, the limit when the turning point touches the horizon, $y_t \to 1$, is not compatible with keeping L finite. In such case the integral develops a singularity that should be canceled with the proper limit for the prefactors, $\lim_{y_t \to 1}(\tilde{\alpha}\tilde{\gamma}) \to 0$. In this case $\tilde{\alpha}\tilde{\gamma} = \frac{\tilde{\alpha} y_t^2}{\sqrt{\tilde{\alpha}^2 - 1}} \to \frac{\tilde{\alpha}}{\sqrt{\tilde{\alpha}^2 - 1}}$ and since $\tilde{\alpha} \geq 1$, this limit can never go to zero, and the cancellation can not occur.

In the configuration $y_t \gg 1$ we obtain

$$
\frac{L}{\beta} \approx \frac{\tilde{\alpha}}{\pi\sqrt{\tilde{\alpha}^2 - 1}} \arccos(\eta^2)\,. \tag{4.32}
$$

Thus each value of $\tilde{\alpha}$ fixes a maximum length, $\frac{L}{\beta} \leq \frac{\tilde{\alpha}}{2\sqrt{\tilde{\alpha}^2 - 1}}$.

The action, within the same condition $y_t \gg 1$, is

$$
S_r \approx \frac{2\lambda\hat{t}}{\beta} \left(\sqrt{\tilde{\alpha}^2 - 1}\,\tan\left(\frac{\pi\sqrt{\tilde{\alpha}^2 - 1}\,L}{\beta\tilde{\alpha}} \right) + \frac{\pi}{\beta}\,\frac{L}{\tilde{\alpha}} \right). \tag{4.33}
$$

We take $\tilde{\gamma}, L, y_b$ as the free parameters. Notice that now it is not possible to keep $v > 1$ while having $V = 1$. To get the lightlike situation we need to send $v \to 1$ and $y_b \to \infty$. These limits can be done now independently. Sending $v \to 1$ from above is equivalent to $\tilde{\gamma} \to \infty$. We will examine the different possibilities.

(a) We take first the $\tilde{\gamma} \to \infty$ limit keeping y_b finite. Since now

$$\frac{y_b^4}{\tilde{\gamma}^2} = \frac{\tilde{\alpha}^2 - 1}{\eta^4} = \frac{\tilde{\alpha}^2 - 1}{\left(\cos\left(\frac{\pi\sqrt{\tilde{\alpha}^2-1}\,L}{\beta\tilde{\alpha}}\right)\right)^2}, \tag{4.34}$$

we end up with a single possibility for $\tilde{\alpha}$, which is $\tilde{\alpha} \to 1$. In addition this entails $y_t \to y_b$, so the string stays on the brane ("*short*" string) as long as y_b is finite and we obtain $S_r = \frac{2\pi\lambda\hat{t}}{\beta^2}L$. We can send next $y_b \to \infty$, but this limit does not affect S_r. In addition to that, as already happened in the $v < 1$ case, the subtraction corresponding to two straight strings from the location of the brane at y_b and the horizon at $y = 1$ becomes infinite when $y_b \to \infty$. Indeed the subtraction term,

$$S_0 = \frac{4\lambda\,\hat{t}}{\beta\,\tilde{\gamma}} \int_1^{y_b} dy y\sqrt{\frac{\tilde{\gamma}^2 + y^4}{y^4 - 1}} \tag{4.35}$$

behaves, for $\tilde{\gamma} \to \infty$ and finite $y_b \gg 1$ as $S_0 \approx \frac{2\lambda\,\hat{t}}{\beta}\log(2y_b^2)$. Thus, again, we can not get a finite renormalized action.

(b) Next we can consider to take first the limit $y_b \to \infty$. Sending (4.34) to infinity can be done essentially in two different ways depending on the actual value of L.

(b.1) Sending $\tilde{\alpha} \to \infty$ whereas $\frac{2L}{\beta} \leq 1$.
When $\frac{2L}{\beta} < 1$, using (4.33), we find in the large $\tilde{\alpha}$ limit that the behavior of the action is $S_r = \frac{2\lambda\hat{t}}{\beta}\,\tilde{\alpha}\,\tan(\frac{\pi\,L}{\beta}) + \mathcal{O}(\frac{1}{\tilde{\alpha}})$. On the other hand, the subtraction term S_0 in the large y_b limit is $S_0 = \frac{4\lambda\hat{t}}{\beta\tilde{\gamma}}\frac{y_b^2}{2} +$ finite. The subtraction cannot cancel the divergences because for large $\tilde{\alpha}$ we obtain from (4.34) that

$$\frac{y_b^2}{\tilde{\gamma}} = \frac{\sqrt{\tilde{\alpha}^2 - 1}}{\cos\left(\frac{\pi\sqrt{\tilde{\alpha}^2-1}\,L}{\beta\tilde{\alpha}}\right)} \approx \frac{\tilde{\alpha}}{\cos\left(\frac{\pi\,L}{\beta}\right)} \tag{4.36}$$

and thus, for large $\tilde{\alpha}$

$$S_r - S_0 = \frac{2\lambda\hat{t}}{\beta}\,\frac{\sin\left(\frac{\pi\,L}{\beta}\right) - 1}{\cos\left(\frac{\pi\,L}{\beta}\right)}\,\tilde{\alpha}, \tag{4.37}$$

which diverges linearly with $\tilde{\alpha}$.

(b.2) The critical case $\frac{L}{\beta} = \frac{\pi}{2}$ is more subtle. In this case, the expansion of (4.33) for large $\tilde{\alpha}$ gives

$$S_r = \frac{2\lambda\hat{t}}{\beta}\left(\frac{4}{\pi}\,\tilde{\alpha}^3 - \frac{3}{\pi}\,\tilde{\alpha} + \mathcal{O}\left(\frac{1}{\tilde{\alpha}}\right)\right). \tag{4.38}$$

On the other hand, using (4.36) and (4.35), and expanding for large $\tilde{\alpha}$, we get

$$S_0 = \frac{2\lambda\hat{t}}{\beta}\left(\frac{4}{\pi}\,\tilde{\alpha}^3 - \frac{3}{\pi}\,\tilde{\alpha} + \mathcal{O}\left(\frac{1}{\tilde{\alpha}}\right)\right) + \text{finite}. \tag{4.39}$$

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Thus the $\tilde{\alpha}$-divergences cancel out and we end up with a finite contribution coming form S_0, as long as $\tilde{\gamma}$ is still kept finite. To isolate this finite contribution it is convenient to consider the subtraction $S_0 -$ divergences,

$$S_0 - \text{divergences} = \frac{2\lambda\hat{t}}{\beta}\int_{\frac{1}{\sqrt{\tilde{\gamma}}}}^{x_b} dx\, x\left(\frac{\sqrt{1+x^4}}{\sqrt{x^4-1/\tilde{\gamma}^2}} - 1\right), \tag{4.40}$$

where we have used the definition $y = \sqrt{\tilde{\gamma}}\, x$.

This integration is finite for $x_b \to \infty$ as long as $\tilde{\gamma}$ is kept finite. But since the last step of our whole procedure must be to send $\tilde{\gamma} \to \infty$, we see that the integration above develops a logarithmic divergence in $\tilde{\gamma}$. Thus it is not possible to obtain a finite renormalized action in the critical case $\frac{L}{\beta} = \frac{\pi}{2}$.

(b.3) A second possible limit is keeping a finite $\tilde{\alpha} > 1$ such that $\cos(\frac{\pi\sqrt{\tilde{\alpha}^2-1}\,L}{\beta\tilde{\alpha}}) = 0$, which is $\frac{2L}{\beta} = \frac{\tilde{\alpha}}{\sqrt{\tilde{\alpha}^2-1}} > 1$.

Using $y_b = 1/\epsilon$, the action becomes, up to an irrelevant factor $\frac{4\lambda\hat{t}}{\beta}$,

$$S_r \approx \frac{1}{\tilde{\gamma}}\int_{\tilde{\gamma}^{1/2}(\tilde{\alpha}^2-1)^{1/4}}^{1/\epsilon} \frac{(y^4+\tilde{\gamma}^2)dy}{y\sqrt{y^4-\tilde{\gamma}^2(\tilde{\alpha}^2-1)}} \approx \frac{1}{2\epsilon^2\tilde{\gamma}^2} + \frac{\pi}{4\sqrt{\tilde{\alpha}^2-1}}. \tag{4.41}$$

In order to evaluate the contribution of the straight strings

$$S_0 = \frac{1}{\tilde{\gamma}}\int_1^{1/\epsilon} \frac{\sqrt{\tilde{\gamma}^2+y^4}\; y\; dy}{\sqrt{y^4-1}}, \tag{4.42}$$

we split the integral at the point $\sqrt{\tilde{\gamma}}(\tilde{\alpha}^2-1)^{1/4}$. The part above this point gives

$$S_0^{\text{above}} \approx \frac{1}{\tilde{\gamma}}\int_{\sqrt{\tilde{\gamma}}(\tilde{\alpha}^2-1)^{1/4}}^{1/\epsilon} \frac{\sqrt{\tilde{\gamma}^2+y^4}\; dy}{y} \approx -\frac{\tilde{\alpha}}{2} + \frac{1}{2\tilde{\gamma}\epsilon^2} + \frac{1}{2}\log\frac{\tilde{\alpha}+1}{\sqrt{\tilde{\alpha}^2-1}}, \tag{4.43}$$

canceling the divergence in S_r. The remaining piece, below that point, is

$$S_0^{\text{below}} \approx \frac{1}{\tilde{\gamma}}\int_1^{\sqrt{\tilde{\gamma}}(\alpha^2-1)^{1/4}} \frac{\tilde{\gamma} y\; dy}{\sqrt{y^4-1}} \approx \frac{\log 2}{2} + \frac{\log\tilde{\gamma}}{2} + \frac{1}{2}\log\sqrt{\tilde{\alpha}^2-1}, \tag{4.44}$$

which becomes divergent when finally $\tilde{\gamma}$ is sent to infinity, thus again we reach the conclusion that the action diverges.

Summing up, except for the "*down*" configurations reaching the horizon, in all the cases there is no sensible way to obtain a finite renormalized action for the Wilson loop, in the limit from a spacelike to a lightlike configuration. The "*down*" configurations, on the other hand, yield a vanishing renormalized action and therefore a zero value for the jet quenching parameter $\hat{q}$, thus confirming the findings in [17].

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